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On the Hamiltonian structure of the spin Ruijsenaars–Schneider model

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Abstract. The Hamiltonian structure of the spin generalization of the rational Ruijsenaars–Schneider model is found by using the Hamiltonian reduction technique. It is shown that the model possesses current algebra symmetry. The possibility of generalizing the obtained Poisson structure to the trigonometric case is discussed and degeneration to the Euler–Calogero–Moser system is examined.

1. Introduction

Recently a spin generalization [1] of the elliptic Ruijsenaars–Schneider model [2, 3] (spin RS model) was introduced as a dynamical system describing the pole evolution of the elliptic solutions of the non-Abelian two-dimensional (2D) Toda chain§. Equations of motion proposed for the model generalize the ones for the Euler–Calogero–Moser (ECM) system [5–9], which is an integrable system of N particles with internal degrees of freedom interacting by a special pairwise potential.

An important tool for dealing with classical integrable systems and especially for quantizing them is the Hamiltonian formalism. Although equations of motion defining the spin RS model can be integrated in terms of Riemann theta-functions, the question about their Hamiltonian form remains open. The aim of the present paper is to give a partial answer to this question, which lies in constructing the explicit Hamiltonian formulation for the rational spin RS model.

Our construction is based on the Hamiltonian reduction procedure acknowledged as the unifying approach to dynamical systems of Calogero or Ruijsenaars type [10–19]. In this approach one starts with a large initial phase space and a simple Hamiltonian possessing a symmetry group. By then factorizing the corresponding motion by this symmetry one is left with a non-trivial dynamical system defined on a reduced phase space. In particular, the rational RS model and the trigonometric Calogero–Moser system appear in this way if one uses the cotangent bundle T^*G over a Lie group G as the initial phase space [12].

A natural generalization of this approach allowing us to include spin variables consists in replacing T^*G by a more general phase space $\mathcal{P}$ that we choose to be $T^*G \times \mathcal{J}^*$, where $\mathcal{J}^*$ is a dual space to the Lie algebra $\mathcal{J}$ of G. Considering on $\mathcal{P}$ a special Hamiltonian H_R and performing the Hamiltonian reduction by G-action, we obtain the Poisson structure of the rational spin RS model.

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§ The trigonometric spin RS model can also be related to affine Toda solitons [4].

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Let us briefly describe the content of the paper and the results obtained. For simplicity, we restrict ourselves to the case of $G = GL(N, \mathbb{C})$. In section 2 we define on $\mathcal{P}$ two dynamical systems governed by Hamiltonians H_C and H_R and show that the corresponding integrals of motion combine into generators of the Yangian, and the current algebra, respectively. Since all these integrals are gauge-invariant, the corresponding symmetries will survive after the reduction.

As is known [14] the dynamical system on the reduced phase space corresponding to H_C is the trigonometric ECM model. This immediately reproduces the result found in [7, 8] that the model possesses Yangian symmetry.

Section 3 is devoted to the rational spin RS model. First, we introduce G-invariant spin variables that after solving the moment map equation can be identified with coordinates on the reduced phase space $\mathcal{P}_r$. Equations of motion for dynamical variables of $\mathcal{P}_r$ produced by H_R coincide with the ones introduced in [1] for the rational case. That is the way we obtain an explicit Hamiltonian formulation of the spin RS model. The Poisson structure of the model is found to be rather non-trivial and admits at least two equivalent descriptions in terms of different phase variables. Moreover, it depends on a parameter γ being a coupling constant of the model.

It turns out that the spin RS model admits a (spectral-independent) L-operator (Lax matrix) that satisfies the same L-operator algebra as does the corresponding spinless model. We also show that the Hamiltonian reduction provides an alternative way of solving equations of motion without using spectral curves. A similar method of integrating equations of motion of the spin RS model is used in [20].

Finally, we present an explicit expression for generators of the current algebra via phase variables of the spin RS model and define the gauge-invariant momentum variables.

In section 4 degeneration of the rational spin RS model to the ECM system is examined. An interesting feature we come across here is the appearance of spin variables obeying the defining relations of the Frobenius Lie algebra. We observe that the general elliptic ECM system can also be formulated in terms of Frobenius spin variables.

2. Current and Yangian symmetries

In this section we construct representations of the Yangian and current algebras related to the cotangent bundle T^*G over the matrix group $G = GL(N, \mathbb{C})$ and describe their connection to the ECM and the spin RS models respectively.

Consider the following manifold $\mathcal{P} = T^*G \times \mathcal{G}^*$, where $\mathcal{G}^*$ is a dual space to the Lie algebra $\mathcal{G} = \mathrm{Mat}(N, \mathbb{C})$ of G. Due to the isomorphism $\mathcal{G}^* \approx \mathcal{G}$ we can parametrize an element from $\mathcal{G}^*$ by a matrix $S \in \mathcal{G}$. The space T^*G is naturally isomorphic to $\mathcal{G}^* \times G$ and we parametrize it by pairs (A, g), where $A \in \mathcal{G}$ and $g \in G$. The algebra of regular functions on $\mathcal{P}$ is supplied with a Poisson structure, which can be written in terms of the variables (A, g, S) as follows

$$\{A_1, A_2\} = \tfrac{1}{2}[C, A_1 - A_2] \tag{2.1}$$

$$\{A_1, g_2\} = g_2 C \qquad \{g_1, g_2\} = 0 \tag{2.2}$$

$$\{S_1, g_2\} = \{S_1, A_2\} = 0 \tag{2.3}$$

$$\{S_1, S_2\} = -\tfrac{1}{2}[C, S_1 - S_2] \tag{2.4}$$

where we use standard tensor notation and $C = \sum_{i,j} E_{ij} \otimes E_{ji}$ is the permutation operator.

The Poisson structure is invariant under the following action of the group G

$$A \to hAh^{-1} \qquad g \to hgh^{-1} \qquad S \to hSh^{-1} \tag{2.5}$$

and so we refer to (2.5) as gauge transformations. The moment map of this action is of the form

$$\mu = gAg^{-1} - A + S. \tag{2.6}$$

The simplest gauge-invariant Hamiltonians are $H_C = \operatorname{tr} A^2$ and $H_R = \operatorname{tr} g$.

The Poisson bracket of the variables S_{ij} can be realized by using $2N$ l-dimensional vectors a_i, b_i which form lN-pairs of canonically conjugated variables

$$\{a_i^\alpha, b_j^\beta\} = -\delta_{ij}\delta^{\alpha\beta}$$

where $i, j = 1, \ldots, N$ and $\alpha, \beta = 1, \ldots, l$. Supposing the matrix elements of S are

$$S_{ij} = \sum_\alpha a_i^\alpha b_j^\alpha \tag{2.7}$$

one recovers the Poisson bracket (2.4). Obviously, under gauge transformations the variables a and b transform in the following way

$$a^\alpha \to ha^\alpha \qquad b^\alpha \to b^\alpha h^{-1}$$

where we regard a^α as a column and b^α as a row.

The variables a and b allow one to construct a lot of gauge-invariants Poisson commuting with H_C or with H_R.

First, we consider a family of integrals of motion for H_C: $I_n^{\alpha\beta} = \operatorname{tr} A^n S^{\alpha\beta}$, where for any α and β the matrix $S^{\alpha\beta}$ has the entries $S_{ij}^{\alpha\beta} = a_i^\alpha b_j^\beta$. In fact, the integrals I_n form a representation of the classical Yangian. To see this, one can introduce the following generating function $T^{\alpha\beta}(z)$ of I_n

$$T^{\alpha\beta}(z) = \delta^{\alpha\beta} + \operatorname{tr} \frac{1}{z-A} S^{\alpha\beta}$$

then, by using the Poisson bracket for the variables $S^{\alpha\beta}$

$$\{S_1^{\alpha\beta}, S_2^{\mu\nu}\} = C_{12}(\delta^{\beta\mu} S_2^{\alpha\nu} - \delta^{\alpha\nu} S_1^{\mu\beta})$$

and performing simple calculations, one obtains the Yangian algebra

$$\{T_1(z), T_2(w)\} = [r(z-w), T_1(z)T_2(w)]. \tag{2.8}$$

Here we regard $T(z)$ as an $(l \times l)$-matrix with entries $T^{\alpha\beta}(z)$, and $r(z-w)$ is a rational solution of the classical Yang–Baxter equation

$$r(z-w) = \frac{K}{z-w}$$

where K is the permutation operator acting in $\mathbb{C}^l \otimes \mathbb{C}^l$.

A well known property of the Yangian is the existence of the involutive subalgebra generated by $I_k(z) = \operatorname{tr} T(z)^k$.

For the Hamiltonian H_R one can choose the following family of integrals of motion $J_n^{\alpha\beta} = \operatorname{tr} g^n S^{\alpha\beta}$. Introducing the formal generating function $J(z)$

$$J^{\alpha\beta}(z) = \sum_{n=-\infty}^{\infty} J_n^{\alpha\beta} z^{-n-1}$$

one can easily show that $J(z)$ satisfies the current algebra relations

$$\{J_1(z), J_2(w)\} = [K, J_2(w)]\delta\left(\frac{z}{w}\right) \tag{2.9}$$

where

$$\delta\left(\frac{z}{w}\right) = \frac{1}{z}\sum_{n=-\infty}^{\infty}\left(\frac{z}{w}\right)^n$$

is the formal δ-function.

It is obvious that $\operatorname{tr} J(z)^n$ are central elements of the current algebra. In addition, the current algebra admits an involutive family of integrals of motion polynomial in g and S. It is constructed as $J_n^+(z) = \operatorname{tr} J^+(z)^n$, where $J^+(z) = \sum_{n=0}^{\infty} J_n z^{-n-1}$. The involutivity is a consequence of the algebra satisfied by $J^+(z)$

$$\{J_1^+(z), J_2^+(w)\} = [r(z-w), J_1^+(z) + J_2^+(w)] \tag{2.10}$$

and it is well known that the Yangian (2.8) is a deformation of (2.10).

The dynamical systems governed by the Hamiltonians H_C and H_R are trivial. However, factorizing the initial phase space $\mathcal{P}$ by the action of a symmetry group, one obtains non-trivial systems defined on the reduced phase space $\mathcal{P}_r$. In particular, to obtain the ECM and the spin RS models one should fix the moment map as [21]

$$gAg^{-1} - A + S = \gamma I \tag{2.11}$$

where γ is a complex number being identified with a coupling constant. Then by solving this equation modulo the action of the gauge group G (G coincides with the isotropy group of (2.11), i.e. (2.11) is a set of first-class constraints), one obtains the reduced phase space. The dynamical systems on $\mathcal{P}_r$ corresponding to the Hamiltonians H_C and H_R are identified with the ECM, and the spin RS models, respectively.

Since the generators of the Yangian and current algebras are gauge-invariant, we conclude that the ECM model possesses Yangian symmetry whereas the spin RS model has current symmetry. As was mentioned in the introduction this result for the ECM model was obtained in [7, 8] by exploiting an explicit L-operator describing the model.

3. Rational spin RS model

In this section we present the Hamiltonian formulation of the spin RS model by considering the reduction of the phase space $\mathcal{P}$ by the action of G. Given the moment map, the space of functions on the reduced phase space $\mathcal{P}_r$ can be identified with the space $\mathrm{Fun}^G\mathcal{P}$ of G-invariant functions on $\mathcal{P}$ restricted to the surface (2.11) of the constant moment level. A choice of an appropriate basis in $\mathrm{Fun}^G\mathcal{P}$ and calculation of the induced Poisson structure make the description of $\mathcal{P}_r$ explicit.

To construct a basis in $\mathrm{Fun}^G\mathcal{P}$ we first note that any semisimple element of $\mathcal{G}$ can be diagonalized by a gauge transformation

$$A = TQT^{-1} \tag{3.12}$$

where Q is a diagonal matrix with gauge-invariant entries $q_i \neq q_j$. By using the action of the Weyl group we fix the order of q_i. For a given A, the matrix T in (3.12) is uniquely defined by being an element of the Frobenius group, i.e. it satisfies the condition

$$Te = e \tag{3.13}$$

where e is an N-dimensional column with all $e_i = 1$. Such a choice for T is known [16] to be relevant for the description of the RS model.

Given A and g, we can diagonalize a matrix $A' = gAg^{-1} = UQU^{-1}$ with the help of an element U such that $Ue = e$. This introduces a useful parametrization for g: $g = UPT^{-1}$, where P is some diagonal matrix.

Under gauge transformation (2.5) matrices T and U transform as follows to $T \to hTh[T]$ and $U \to hUh[U]$ where $h[T]$ and $h[U]$ are the diagonal matrices $h[T]_i = (T^{-1}h^{-1}e)_i$ and $h[U]_i = (U^{-1}h^{-1}e)_i$.

We next introduce the diagonal matrices $t_{ij} = t_i\delta_{ij}$ and $u_{ij} = u_i\delta_{ij}$ with entries

$$t_i = \sum_\alpha (T^{-1}a^\alpha)_i \qquad u_i = \sum_\alpha (U^{-1}a^\alpha)_i \tag{3.14}$$

which transform under gauge transformation (2.5) in the following way

$$t_i \to h[T]_i^{-1}t_i \qquad u_i \to h[U]_i^{-1}u_i$$

and we use t to define the G-invariant spin variables

$$\boldsymbol{a}_i^\alpha = t_i^{-1}(T^{-1}a^\alpha)_i \qquad \boldsymbol{c}_i^\alpha = t_i(b^\alpha UP)_i.$$

Note that $\boldsymbol{a}_i^\alpha$ are not arbitrary but satisfy the constraints $\sum_\alpha \boldsymbol{a}_i^\alpha = 1$ for any i. The relevance of this definition will be clarified later.

To calculate the Poisson algebra of $\boldsymbol{a}$ and $\boldsymbol{c}$ one needs to use the one for (T, U, P, Q)-variables. In [16] it was proved that the standard Poisson structure (2.1) and (2.2) on T^*G rewritten in terms of (T, U, P, Q)-variables has the form

$$\{T_1, T_2\} = T_1T_2r_{12} \qquad \{U_1, U_2\} = -U_1U_2r_{12} \tag{3.15}$$

$$\{T_1, P_2\} = T_1P_2\bar{r}_{12} \qquad \{U_1, P_2\} = U_1P_2\bar{r}_{12} \tag{3.16}$$

$$\{T_1, Q_2\} = \{U_1, Q_2\} = \{P_1, P_2\} = \{T_1, U_2\} = 0 \tag{3.17}$$

$$\{Q_1, Q_2\} = 0 \qquad \{Q_1, P_2\} = P_2\sum_i E_{ii} \otimes E_{ii}. \tag{3.18}$$

Here r_{12} is an N-parametric solution of the classical Yang–Baxter equation

$$r_{12} = \sum_{i\neq j} \frac{1}{q_{ij}} F_{ij} \otimes F_{ji} \tag{3.19}$$

where $F_{ij} = E_{ii} - E_{ij}$ is a basis of the Frobenius Lie algebra and the matrix $\bar{r}_{12}$ is given by

$$\bar{r}_{12} = \sum_{i\neq j} \frac{1}{q_{ij}} F_{ij} \otimes E_{jj}. \tag{3.20}$$

Note that (3.18) implies that q_i and $p_i = \log P_i$ are canonically conjugated variables.

With formulae (3.15)–(3.18) to hand we first calculate

$$\{t_i, t_j\} = -\frac{1}{q_{ij}}(t_i - t_j)^2 \qquad \{u_i, u_j\} = \frac{1}{q_{ij}}(u_i - u_j)^2 \qquad \{t_i, u_j\} = 0$$

and then the Poisson brackets of the invariant spins

$$\{\boldsymbol{a}_i^\alpha, \boldsymbol{a}_j^\beta\} = \frac{1}{q_{ij}}(\boldsymbol{a}_i^\alpha\boldsymbol{a}_j^\beta + \boldsymbol{a}_j^\alpha\boldsymbol{a}_i^\beta - \boldsymbol{a}_i^\alpha\boldsymbol{a}_i^\beta - \boldsymbol{a}_j^\alpha\boldsymbol{a}_j^\beta) \tag{3.21}$$

$$\{\boldsymbol{c}_i^\alpha, \boldsymbol{c}_j^\beta\} = \frac{1}{q_{ij}}(\boldsymbol{c}_i^\alpha\boldsymbol{c}_j^\beta + \boldsymbol{c}_j^\alpha\boldsymbol{c}_i^\beta) + \boldsymbol{c}_j^\beta\boldsymbol{L}_{ji} - \boldsymbol{L}_{ij}\boldsymbol{c}_i^\alpha \tag{3.22}$$

$$\{\boldsymbol{c}_i^\alpha, \boldsymbol{a}_j^\beta\} = \delta^{\alpha\beta}\boldsymbol{L}_{ji} - \boldsymbol{a}_j^\beta\boldsymbol{L}_{ji} + \frac{1}{q_{ij}}\boldsymbol{c}_i^\alpha(\boldsymbol{a}_i^\beta - \boldsymbol{a}_j^\beta) \tag{3.23}$$

$$\{\boldsymbol{a}_i^\alpha, \boldsymbol{c}_j^\beta\} = -\delta^{\alpha\beta}\boldsymbol{L}_{ij} + \boldsymbol{a}_i^\alpha\boldsymbol{L}_{ij} + \frac{1}{q_{ij}}\boldsymbol{c}_j^\beta(\boldsymbol{a}_j^\alpha - \boldsymbol{a}_i^\alpha). \tag{3.24}$$

The Poisson structure of invariant spins is not closed since it involves another gauge invariant object $\boldsymbol{L} : \boldsymbol{L}_{ij} = t_i^{-1}L_{ij}t_j$, where $L = T^{-1}gT$. In [16] L was identified with the L-operator of the rational RS model. Analogously, $\boldsymbol{L}$ will be called the L-operator of the spin RS model. The relevance of this definition will be justified later. Note that in equations (3.21)–(3.24) and in later formulae it is assumed that if some denominator becomes zero, the corresponding fraction is omitted.

Calculating the Poisson algebra for $\boldsymbol{L}$ with the help of equations (3.15)–(3.18), we obtain that it coincides with the one for L, namely

$$\{\boldsymbol{L}_1, \boldsymbol{L}_2\} = r_{12}\boldsymbol{L}_1\boldsymbol{L}_2 + \boldsymbol{L}_1\boldsymbol{L}_2\hat{r}_{12} + \boldsymbol{L}_1\bar{r}_{21}\boldsymbol{L}_2 - \boldsymbol{L}_2\bar{r}_{12}\boldsymbol{L}_1 \tag{3.25}$$

where $\hat{r}_{12} = \bar{r}_{12} - \bar{r}_{21} - r_{12}$ is a constant solution of the Gervais–Neveu–Felder equation [21, 22]. Therefore, the L-operator algebra for the spin RS model and the one for the RS model without spins are the same.

To complete the description of the Poisson algebra of invariant spins we find the Poisson brackets of $\boldsymbol{L}$ with $\boldsymbol{a}$ and $\boldsymbol{c}$:

$$\{a_i^\alpha, \boldsymbol{L}_{kl}\} = \frac{1}{q_{ik}}(a_i^\alpha - a_k^\alpha)\boldsymbol{L}_{kl} - \frac{1}{q_{ik}}(a_i^\alpha - a_k^\alpha)\boldsymbol{L}_{il} - \frac{1}{q_{il}}(a_i^\alpha - a_l^\alpha)\boldsymbol{L}_{kl} \tag{3.26}$$

$$\{c_i^\alpha, \boldsymbol{L}_{kl}\} = \frac{1}{q_{il}}c_i^\alpha\boldsymbol{L}_{kl} + \frac{1}{q_{il}}c_l^\alpha\boldsymbol{L}_{ki} - \frac{1}{q_{ik}}c_i^\alpha\boldsymbol{L}_{kl} + \frac{1}{q_{ik}}c_i^\alpha\boldsymbol{L}_{il} + \boldsymbol{L}_{li}\boldsymbol{L}_{kl} - \boldsymbol{L}_{ki}\boldsymbol{L}_{kl}. \tag{3.27}$$

Thus, the Poisson algebra of gauge-invariant variables $\boldsymbol{a}, \boldsymbol{c}$ and $\boldsymbol{L}$ is closed.

We remark that the choice of gauge-invariant spins and the L-operator is not unique. In particular, one could use $\omega_i = \sum_\alpha (b^\alpha T)_i$ to define other gauge-invariant spins $\hat{a}_i^\alpha = \omega_i(T^{-1}a^\alpha)_i$, $\hat{c}_i^\alpha = \omega_i^{-1}(b^\alpha UP)_i$ and the L-operator: $\hat{L}_{ij} = \omega_i L_{ij}\omega_j^{-1}$. One can verify that this set of gauge-invariant variables satisfies a different algebra.

The next step consists in restricting the Poisson algebra (3.21)–(3.27) to the surface (2.11) of the constant moment level. Diagonalizing the variable A, we find that equation (2.11) is equivalent to

$$LQ - QL - \gamma L = -T^{-1}STL. \tag{3.28}$$

Multiplying (3.28) by the diagonal matrix t from the right and by t^{-1} from the left, and taking into account that $L = T^{-1}gT = T^{-1}UP$, we rewrite (3.28) in terms of gauge-invariant variables

$$\boldsymbol{L}Q - Q\boldsymbol{L} - \gamma\boldsymbol{L} = -t^{-1}T^{-1}SUPt \tag{3.29}$$

since $(t^{-1}T^{-1}SUPt)_{ij} = \sum_\alpha \boldsymbol{a}_i^\alpha \boldsymbol{c}_j^\alpha$. Thus, we can solve equation (3.29) with respect to $\boldsymbol{L}$. The solution is given by

$$\boldsymbol{L} = \sum_{ij} \frac{f_{ij}}{q_{ij} + \gamma} E_{ij} \tag{3.30}$$

where we introduce $f_{ij} = \sum_\alpha \boldsymbol{a}_i^\alpha \boldsymbol{c}_j^\alpha$. Now the reduction of the Poisson structure (3.21)–(3.27) on the surface of (3.28) amounts to the substitution on the right-hand side of (3.21)–(3.27) of the entries of the L-operator (3.30). The consistency of the reduced Poisson structure can also be checked by direct calculations. To this end one must first find the Poisson algebra of f_{ij}-variables

$$\begin{aligned}\{f_{ij}, f_{kl}\} = {} & \left(\frac{1}{q_{ik}} + \frac{1}{q_{jl}} + \frac{1}{q_{kj}} + \frac{1}{q_{li}}\right) f_{ij} f_{kl} + \left(\frac{1}{q_{ki}} + \frac{1}{q_{il} + \gamma}\right) f_{ij} f_{il} \\ & + \left(\frac{1}{q_{ik}} + \frac{1}{q_{jl}} + \frac{1}{q_{kj} + \gamma} - \frac{1}{q_{il} + \gamma}\right) f_{il} f_{kj} + \left(\frac{1}{q_{jk}} - \frac{1}{q_{jl} + \gamma}\right) f_{ij} f_{jl} \\ & + \left(\frac{1}{q_{ki}} - \frac{1}{q_{kj} + \gamma}\right) f_{kj} f_{kl} + \left(\frac{1}{q_{il}} + \frac{1}{q_{lj} + \gamma}\right) f_{lj} f_{kl}\end{aligned} \tag{3.31}$$

and $\{f_{ij}, q_k\} = -f_{ij}\delta_{kj}$. Then by using the representation (3.30) for $\boldsymbol{L}$ one does recover the L-operator algebra (3.25).

Now we proceed with describing the dynamics on $\mathcal{P}_r$. The invariant Hamiltonian H_R acquires on $\mathcal{P}_r$ a form $H_R = \mathrm{tr}\,\boldsymbol{L} = (1/\gamma)\sum_i f_{ii}$. This Hamiltonian and the Poisson structure on $\mathcal{P}_r$ produce the following equations of motion

$$\dot{q}_i = L_{ii} = \frac{1}{\gamma} f_{ii} \tag{3.32}$$

$$\dot{\boldsymbol{a}}_i^\alpha = -\sum_{j\neq i} \frac{1}{q_{ij}}(\boldsymbol{a}_i^\alpha - \boldsymbol{a}_j^\alpha)\boldsymbol{L}_{ij} = -\frac{1}{\gamma}\sum_{j\neq i}(\boldsymbol{a}_i^\alpha - \boldsymbol{a}_j^\alpha) f_{ij} V(q_{ij}) \tag{3.33}$$

$$\dot{\boldsymbol{c}}_i^\alpha = \sum_{j\neq i} \frac{1}{q_{ij}}(\boldsymbol{c}_i^\alpha \boldsymbol{L}_{ij} + \boldsymbol{c}_j^\alpha \boldsymbol{L}_{ji}) = \frac{1}{\gamma}\sum_{j\neq i}(\boldsymbol{c}_i^\alpha f_{ij} V(q_{ij}) - \boldsymbol{c}_j^\alpha f_{ji} V(q_{ji})) \tag{3.34}$$

where we introduce the potential $V(q_{ij}) = 1/(q_{ij}) - 1/(q_{ij}+\gamma)$. Differentiating $\dot{q}_i$ and taking into account equations (3.33) and (3.34), one gets

$$\ddot{q}_i = 2\sum_{j\neq i}\frac{1}{q_{ij}} L_{ij}L_{ji} = 2\sum_{j\neq i}\frac{1}{q_{ij}}\boldsymbol{L}_{ij}\boldsymbol{L}_{ji} = \frac{1}{\gamma^2}\sum_{j\neq i} f_{ij}f_{ji}(V(q_{ij}) - V(q_{ji})) \tag{3.35}$$

and equations of motion for f_{ij}

$$\dot{f}_{ij} = \frac{1}{\gamma}\sum_{k\neq i,j} V(q_{kj}) f_{ik} f_{kj} - V(q_{ik}) f_{ik} f_{kj} + V(q_{ik}) f_{ik} f_{ij} - V(q_{jk}) f_{jk} f_{ij}. \tag{3.36}$$

It follows from equations (3.32) and (3.36) that the equation of motion for $\boldsymbol{L}$ can be written in the Lax form $\dot{\boldsymbol{L}} = [\boldsymbol{L}, \boldsymbol{M}]$ with $\boldsymbol{M} = \sum_{i\neq j}(1/q_{ij})\boldsymbol{L}_{ji}F_{ij}$. However, this equation is not equivalent to equations (3.32) and (3.36).

In paper [1] the spin generalization of the elliptic RS model was introduced. The generalized model is a system of N particles with coordinates q_i, each particle having internal degrees of freedom described by the l-dimensional vector a_i^α and the l-dimensional vector c_i^α. The equations of motion generalize the ones for the ECM system

$$\ddot{q}_i = \sum_{j\neq i} f_{ij}f_{ji}(V(q_{ij}) - V(q_{ji})) \tag{3.37}$$

$$\dot{a}_i = \sum_{j\neq i} a_j f_{ij} V(q_{ij}) - \lambda_i a_i \tag{3.38}$$

$$\dot{c}_i = -\sum_{j\neq i} c_j f_{ji} V(q_{ji}) + \lambda_i c_i \tag{3.39}$$

where $V(q) = \zeta(q) - \zeta(q+\gamma)$ and $\lambda_i(t)$ are arbitrary functions of t and $\zeta(q)$ denotes the Weierstrass zeta-function. Equations of motion (3.37)–(3.39) are invariant under rescaling

$$a_i \to k_i a_i \qquad c_i \to \frac{1}{k_i} c_i.$$

Introducing the invariant variables $\hat{a}_i^\alpha = (\sum_\alpha a_i^\alpha)^{-1} a_i^\alpha$ and $\hat{c}_i^\alpha = (\sum_\alpha a_i^\alpha) c_i^\alpha$, and calculating from (3.38) and (3.39) the equations of motion for $\hat{a}_i^\alpha$ and $\hat{c}_i^\alpha$, one discovers that all λ_i drop out and the equations of motion coincide with (3.33) and (3.34) with the change $\hat{a}_i^\alpha \to \boldsymbol{a}_i^\alpha$, $\hat{c}_i^\alpha \to (1/\gamma)\boldsymbol{c}_i^\alpha$ and with substitution of $V(q)$ for its rational analogue $(1/q - 1/(q+\gamma))$. To present equations (3.37) in the Lax form, in paper [1] a spectral-dependent L-operator $L(z)$ was suggested. One can see that in the rational case $L(z)$ coincides with $\boldsymbol{L}$ in the limit $z \to \infty$. Thus, we obtain the Hamiltonian formulation of the spin generalization of the rational RS model.

Now we show how the equations of motion (3.35)–(3.34) can be solved in terms of the factorization problem (see also [2, 11, 20]). The Hamiltonian H_R induces on $\mathcal{P}$ equations of motion

$$\dot{g} = 0 \qquad \dot{A} = g \qquad \dot{S} = 0$$

that can be easily integrated; $A(t) = gt + A_0$, $g(t) =$ constant and $S(t) =$ constant. We suppose that the positions of particles at $t = 0$ are given by q_i lying on $\mathcal{P}_r$. It means that $A(t) = gt + Q$. Since for any t the point $(A(t), g(t), S(t))$ satisfies the constraint (2.11) one finds that g should be identified with the L-operator L_0 at $t = 0$.

Let us now show that the solution of (3.35) is given by the diagonal factor $Q(t)$ in the decomposition of $A(t)$

$$A(t) = L_0 t + Q = T(t)Q(t)T^{-1}(t) \qquad T(t)e = e. \tag{3.40}$$

In [16] it is proved that

$$\frac{\delta T_{ij}}{\delta A_{mn}} = \sum_{a\neq j} \frac{1}{q_{ja}}(T_{ia}T_{nj}T^{-1}_{am} + T_{ij}T_{na}T^{-1}_{jm}) \tag{3.41}$$

$$\frac{\delta q_i}{\delta A_{mn}} = T_{ni}T^{-1}_{im}. \tag{3.42}$$

Using these formulae, we find

$$\dot{q}_i = \sum_{mn} \frac{\delta q_i}{\delta A_{mn}} \frac{dA_{mn}}{dt} = (T^{-1}gT)_{ii}(t) = L_{ii}(t)$$

and differentiating $\dot{q}_i$ once again, one gets

$$\ddot{q}_i = \dot{L}_{ii} = [T^{-1}gT, T^{-1}\dot{T}]_{ii} \tag{3.43}$$

where

$$(T^{-1}\dot{T})_{ij} = -\frac{1}{q_{ij}}L_{ij} + \delta_{ij}\sum_{a\neq j}\frac{1}{q_{ja}}L_{ja}. \tag{3.44}$$

Substituting equation (3.44) into equation (3.43), we obtain equation (3.35).

As to the spin variables their equations of motion are automatically solved if one knows the factor $T(t)$ in the decomposition (3.40). Indeed, if we define $\tilde{a}_i^\alpha(t) = (T^{-1}(t)a^\alpha)_i$ then

$$\dot{\tilde{a}}_i^\alpha = -\sum_{j\neq i}\frac{1}{q_{ij}}(\tilde{a}_i^\alpha - \tilde{a}_j^\alpha)L_{ij}$$

and for the invariant spin $\boldsymbol{a}_i^\alpha = (1/t_i)\tilde{a}_i^\alpha$ we obtain equation (3.33). Solution of the equation of motion for $\boldsymbol{c}_i^\alpha$ is given by $\boldsymbol{c}_i^\alpha(t) = t_i(t)(b^\alpha L_0 T(t))_i$.

The integrals of motion $J_n^{\alpha\beta} = \mathrm{tr}(g^n S^{\alpha\beta})$ introduced in section 2 take on $\mathcal{P}_r$ the following form

$$J_n^{\alpha\beta} = \sum_{ij}(\boldsymbol{L}^{n-1})_{ij}\boldsymbol{a}_j^\alpha \boldsymbol{c}_i^\beta.$$

Substituting here the explicit form (3.30) of the L-operator, one can recast $J_n^{\alpha\beta}$ for $n \geqslant 1$ in the form

$$J_1^{\alpha\beta} = \sum_i S_i^{\beta\alpha}$$

$$J_n^{\alpha\beta} = \sum_{i_1,\ldots,i_n} \frac{(S_{i_1}S_{i_2}\ldots S_{i_n})^{\beta\alpha}}{(q_{i_1i_2}+\gamma)(q_{i_2i_3}+\gamma)\ldots(q_{i_{n-1}i_n}+\gamma)} \tag{3.45}$$

where we use $(l \times l)$-matrices $S_i^{\alpha\beta} = \boldsymbol{c}_i^\alpha \boldsymbol{a}_i^\beta (i = 1, \ldots, N)$.

An important property of S_i-variables is that they form a set of gauge-invariant variables equivalent to $(\boldsymbol{a}, \boldsymbol{c})$. In fact, one can see that

$$\boldsymbol{c}_i^\alpha = \sum_\beta S_i^{\alpha\beta} \qquad \boldsymbol{a}_i^\alpha = \frac{S_i^{\beta\alpha}}{\sum_\gamma S_i^{\beta\gamma}}.$$

The Poisson structure of the model can be conveniently rewritten in terms of $S_i^{\alpha\beta}$

$$\{S_i^{\alpha\beta}, S_j^{\mu\nu}\} = \frac{1}{q_{ij}}(S_i^{\mu\beta} S_j^{\alpha\nu} + S_i^{\alpha\nu} S_j^{\mu\beta}) - \frac{\delta^{\beta\mu}}{q_{ij}+\gamma}(S_i S_j)^{\alpha\nu} + \frac{\delta^{\alpha\nu}}{q_{ji}+\gamma}(S_j S_i)^{\mu\beta}$$
$$\{q_i, S_j^{\alpha\beta}\} = S_j^{\alpha\beta}\delta_{ij}. \tag{3.46}$$

Since the Hamiltonian H_R can be expressed as $H_R = \sum_i \operatorname{Tr} S_i$, (3.35) acquires the form

$$\ddot{q}_i = \frac{1}{\gamma^2}\sum_{j\neq i} \operatorname{Tr}(S_i S_j)(V(q_{ij}) - V(q_{ji})) \tag{3.47}$$

where Tr is used to denote the trace of an $(l \times l)$-matrix. Analogously, equations (3.33) and (3.34) produce the equations of motion for S_i

$$\dot{S}_i = \frac{1}{\gamma}\sum_{j\neq i}(S_i S_j V(q_{ij}) - S_j S_i V(q_{ji})). \tag{3.48}$$

We observe now that the Poisson structure (3.46) and the Hamiltonian H_R are invariant under the transformations $S_i \to \Omega^{-1} S_i \Omega$, where $\Omega \in GL(l, \mathbb{C})$. These transformations are generated by $J_0^{\alpha\beta}$.

Thus, we see that the Hamiltonian formalism of the rational spin RS model can be equivalently presented in terms of either $(\boldsymbol{a}, \boldsymbol{c})$- or S_i-variables. The definition of $\boldsymbol{c}_i^\alpha = t_i(b^\alpha U P)_i$ implies that they contain the variables conjugated to q_i. However, we cannot identify them with P_i since the latter are not gauge-invariant. The gauge-invariant momentum $\boldsymbol{P}_i$ can be defined as $\boldsymbol{P}_i = u_i^{-1} P_i t_i$. Computing the Poisson brackets of $\boldsymbol{P}_i$, one gets that $\{\boldsymbol{P}_i, \boldsymbol{P}_j\} = 0$ and $\{q_i, \boldsymbol{P}_j\} = \delta_{ij}\boldsymbol{P}_j$.

Recalling that the invariant L-operator has the form $\boldsymbol{L} = t^{-1}T^{-1}UPt$, it can be written as $\boldsymbol{L} = \boldsymbol{W}\boldsymbol{P}$, where $\boldsymbol{W}$ is a gauge-invariant variable

$$\boldsymbol{W} = t^{-1}T^{-1}Uu.$$

Then it is easy to see that $\boldsymbol{W}$ belongs to the Frobenius group, i.e. it obeys the condition $\boldsymbol{W}e = e$

$$(\boldsymbol{W}e)_i = \sum_{k,m}(t^{-1}T^{-1})_{ik}U_{km}(U^{-1}a^\alpha)_m = \frac{1}{t_i}(T^{-1}a^\alpha)_i = 1. \tag{3.49}$$

Just as it was for the spinless RS model, the Poisson bracket for $\boldsymbol{W}$ coincides with the Sklyanin bracket

$$\{\boldsymbol{W}_1, \boldsymbol{W}_2\} = [r_{12}, \boldsymbol{W}_1\boldsymbol{W}_2]. \tag{3.50}$$

On $\mathcal{P}_r$ the variable $\boldsymbol{W}$ acquires the form

$$\boldsymbol{W}_{ij} = \sum_\alpha \frac{\boldsymbol{a}_i^\alpha \boldsymbol{b}_j^\alpha}{q_{ij}+\gamma} \tag{3.51}$$

where $\boldsymbol{b}_i^\alpha = \boldsymbol{c}_i^\alpha \boldsymbol{P}_i^{-1}$.

Since the variables $\boldsymbol{a}_i^\alpha$ obey the constraints $\sum_\alpha \boldsymbol{a}_i^\alpha = 1$ for any i, condition (3.49) implies that $\boldsymbol{b}_i^\alpha$ are also not arbitrary but subject to the constraints

$$\sum_\alpha \boldsymbol{a}_i^\alpha \sum_j \frac{\boldsymbol{b}_j^\alpha}{q_{ij}+\gamma} = 1 \tag{3.52}$$

for any i. Therefore, the number of independent spin variables is $2N(l-1)$. In terms of these variables the Poisson structure of $\mathcal{P}_r$ looks as follows

$$\begin{aligned}
&\{q_i, \boldsymbol{P}_j\} = \delta_{ij}\boldsymbol{P}_j \qquad \{q_i, \boldsymbol{a}_j^\alpha\} = 0 = \{q_i, \boldsymbol{b}_j^\alpha\} \\
&\{\boldsymbol{P}_i, \boldsymbol{a}_j^\alpha\} = \frac{1}{q_{ij}}(\boldsymbol{a}_i^\alpha - \boldsymbol{a}_j^\alpha)\boldsymbol{P}_i \\
&\{\boldsymbol{P}_i, \boldsymbol{b}_j^\alpha\} = \left(\delta_{ij} - \boldsymbol{W}_{ij} + \frac{1}{q_{ij}}\boldsymbol{b}_j^\alpha + \delta_{ij}\sum_{n\neq i}\frac{1}{q_{nj}}\boldsymbol{b}_n^\alpha\right)\boldsymbol{P}_i \\
&\{\boldsymbol{a}_i^\alpha, \boldsymbol{a}_j^\beta\} = \frac{1}{q_{ij}}(\boldsymbol{a}_i^\alpha\boldsymbol{a}_j^\beta + \boldsymbol{a}_j^\alpha\boldsymbol{a}_i^\beta - \boldsymbol{a}_i^\alpha\boldsymbol{a}_i^\beta - \boldsymbol{a}_j^\alpha\boldsymbol{a}_j^\beta) \\
&\{\boldsymbol{b}_i^\alpha, \boldsymbol{b}_j^\beta\} = \delta_{ij}(\boldsymbol{b}_i^\beta - \boldsymbol{b}_i^\alpha) + \frac{1}{q_{ij}}(\boldsymbol{b}_j^\alpha\boldsymbol{b}_i^\beta - \boldsymbol{b}_j^\beta\boldsymbol{b}_i^\alpha) + \delta_{ij}\sum_{n\neq i}\frac{1}{q_{in}}(\boldsymbol{b}_n^\beta\boldsymbol{b}_i^\alpha - \boldsymbol{b}_n^\alpha\boldsymbol{b}_i^\beta) \\
&\{\boldsymbol{a}_i^\alpha, \boldsymbol{b}_j^\beta\} = -\delta^{\alpha\beta}\boldsymbol{W}_{ij} + \boldsymbol{a}_i^\alpha\boldsymbol{W}_{ij}
\end{aligned} \tag{3.53}$$

where $\boldsymbol{W}$ is given by (3.51). One should point out that the structure (3.53) is Poisson only due to the constraints imposed on the spin variables. Therefore, the rational spin RS model provides a new realization of the Poisson relations (3.50) as well as the L-operator algebra (3.25).

Now we discuss the problem of generalizing the found Poisson structure for the spin rational RS model to the trigonometric case.

Relying on the fact that in both spin and spinless cases the L-operator algebras may be the same, one can easily derive the trigonometric analogue of the Poisson bracket (3.31) for the variables f_{ij}. It follows from the results of [17] that the trigonometric RS model can be described by the L-operator algebra (3.25), where this time the r-matrices r, $\bar{r}$ and $\hat{r}$ are given by

$$\begin{aligned}
r = {}&\sum_{ij} E_{ij}\otimes E_{ji} + \sum_{i\neq j}\coth(q_{ij})E_{ii}\otimes E_{jj} + \sum_{i\neq j}\coth(q_{ij})E_{ij}\otimes E_{ji} \\
&- \sum_{i\neq j}\frac{\mathrm{e}^{q_{ij}}}{\sinh(q_{ij})}E_{ij}\otimes E_{jj} + \sum_{i\neq j}\frac{\mathrm{e}^{q_{ij}}}{\sinh(q_{ij})}E_{jj}\otimes E_{ij}
\end{aligned} \tag{3.54}$$

$$\bar{r} = -\sum_i E_{ii}\otimes E_{ii} + \sum_{i\neq j}\coth(q_{ij})E_{ii}\otimes E_{jj} - \sum_{i\neq j}\frac{\mathrm{e}^{q_{ij}}}{\sinh(q_{ij})}E_{ij}\otimes E_{jj} \tag{3.55}$$

$$\hat{r} = -\sum_{ij} E_{ij}\otimes E_{ji} + \sum_{i\neq j}\coth(q_{ij})(E_{ii}\otimes E_{jj} - E_{ij}\otimes E_{ji}). \tag{3.56}$$

For the spinless trigonometric RS model the L-operator satisfying (3.25) is of the form

$$L = \sum_{ij}\frac{\mathrm{e}^{q_{ij}+\gamma}}{\sinh(q_{ij}+\gamma)}c_j E_{ij}$$

where c_j are some functions of dynamical variables. It is natural to assume that in the spin case the corresponding L-operator has the form

$$L = \sum_{ij}\frac{\mathrm{e}^{q_{ij}+\gamma}}{\sinh(q_{ij}+\gamma)}f_{ij}E_{ij}$$

then the Poisson relations for f_{ij} follow immediately from the L-operator algebra (3.25)

$$\begin{aligned}\{f_{ij}, f_{kl}\} = &(\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}) + \coth(q_{li}))f_{ij}f_{kl}\\ &+(\coth(q_{ik}) + \coth(q_{jl}) + \coth(q_{kj}+\gamma) - \coth(q_{il}+\gamma))f_{il}f_{kj}\\ &+(\coth(q_{ki}) + \coth(q_{il}+\gamma))f_{ij}f_{il} + (\coth(q_{jk}) - \coth(q_{jl}+\gamma))f_{ij}f_{jl}\\ &+(\coth(q_{ki}) - \coth(q_{kj}+\gamma))f_{kj}f_{kl} + (\coth(q_{il}) + \coth(q_{lj}+\gamma))f_{lj}f_{kl}\end{aligned} \quad (3.57)$$

and they look like a trigonometric generalization of (3.31).

Now one can easily verify that the equations of motion for f_{ij} are given by (3.36) with the potential $V(q) = \coth(q) - \coth(q+\gamma)$ and with a change of the overall factor $1/\gamma$ by $e^{\gamma}/(\sinh(\gamma))$. On the other hand, these equations follow from equations (3.38) and (3.39). The problem of describing the Poisson structure of the trigonometric spin RS model could be completely solved if one found such Poisson brackets for the variables a and c that could induce the ones (3.57) for f_{ij}. A straightforward generalization of equations (3.21)–(3.24) to the case at hand by replacing $1/(q_{ij})$ by $\coth(q_{ij})$ fails in this regard so at the moment we cannot offer a solution of the problem.

4. Euler–Calogero–Moser model

We start this section by discussing the degeneration of the spin RS system to the rational ECM model. For this purpose we rescale $\log \boldsymbol{P}_i = \boldsymbol{p}_i \to \varepsilon \boldsymbol{p}_i$ and $q_i \to (1/\varepsilon)q_i$, and consider the limit $\varepsilon \to 0$. The constraint (3.49) implies that in this limit

$$\sum_{\alpha} \boldsymbol{a}_i^{\alpha}\boldsymbol{b}_i^{\alpha} = \boldsymbol{S}_{ii} = \gamma \quad (4.58)$$

and $\boldsymbol{W}_{ij}$ has the following expansion

$$\boldsymbol{W}_{ij} = \delta_{ij} + \varepsilon(1-\delta_{ij})\frac{\boldsymbol{S}_{ij}}{q_{ij}} - \varepsilon\delta_{ij}\sum_{k\neq i}\frac{\boldsymbol{S}_{ik}}{q_{ik}} + o(\varepsilon)$$

where $\boldsymbol{S}_{ij} = \sum_{\alpha} \boldsymbol{a}_i^{\alpha}\boldsymbol{b}_j^{\alpha}$. The corresponding expansion of the L-operator produces in the first-order in ε the L-operator $\mathcal{L}$ of the rational ECM model

$$\mathcal{L}_{ij} = \delta_{ij}\left(\boldsymbol{p}_i - \sum_{k\neq i}\frac{\boldsymbol{S}_{ik}}{q_{ik}}\right) + (1-\delta_{ij})\frac{\boldsymbol{S}_{ij}}{q_{ij}}. \quad (4.59)$$

In the limit $\varepsilon \to 0$ the Poisson structure (3.53) reduces to

$$\begin{aligned}&\{q_i, \boldsymbol{p}_j\} = \delta_{ij} \qquad \{q_i, \boldsymbol{a}_j^{\alpha}\} = 0 = \{q_i, \boldsymbol{b}_j^{\alpha}\}\\ &\{\boldsymbol{p}_i, \boldsymbol{a}_j^{\alpha}\} = \frac{1}{q_{ij}}(\boldsymbol{a}_i^{\alpha} - \boldsymbol{a}_j^{\alpha})\\ &\{\boldsymbol{p}_i, \boldsymbol{b}_j^{\alpha}\} = \delta_{ij}\sum_{k\neq i}\frac{\boldsymbol{S}_{ik}}{q_{ik}} - (1-\delta_{ij})\frac{\boldsymbol{S}_{ij}}{q_{ij}} + \frac{1}{q_{ij}}\boldsymbol{b}_j^{\alpha} + \delta_{ij}\sum_{n\neq i}\frac{1}{q_{nj}}\boldsymbol{b}_n^{\alpha}\\ &\{\boldsymbol{a}_i^{\alpha}, \boldsymbol{a}_j^{\beta}\} = 0\\ &\{\boldsymbol{b}_i^{\alpha}, \boldsymbol{b}_j^{\beta}\} = \delta_{ij}(\boldsymbol{b}_i^{\beta} - \boldsymbol{b}_i^{\alpha})\\ &\{\boldsymbol{a}_i^{\alpha}, \boldsymbol{b}_j^{\beta}\} = -\delta^{\alpha\beta}\delta_{ij} + \boldsymbol{a}_i^{\alpha}\delta_{ij}\end{aligned} \quad (4.60)$$

and by introducing new momenta

$$p_i = \boldsymbol{p}_i - \sum_{k\neq i} \frac{\boldsymbol{S}_{ik}}{q_{ik}}$$

one can check that they have vanishing Poisson brackets with $\boldsymbol{a}_i^\alpha$ and $\boldsymbol{b}_i^\alpha$. The L-operator $\mathcal{L}$ turns into the standard one used in the description of the ECM system [5–8]

$$\mathcal{L}_{ij} = \delta_{ij} p_i + (1-\delta_{ij}) \frac{\boldsymbol{S}_{ij}}{q_{ij}}. \tag{4.61}$$

To make a contact with the usual description of the ECM system we introduce lN pairs of canonical variables: $\{a_i^\alpha, b_j^\beta\} = -\delta_{ij}\delta^{\alpha\beta}$. Then the invariant variables $\boldsymbol{a}_i^\alpha$ and $\boldsymbol{b}_j^\beta$ with the Poisson algebra (4.60) can be realized as

$$\boldsymbol{a}_i^\alpha = \frac{a_i^\alpha}{\sum_\beta a_i^\beta} \qquad \boldsymbol{b}_i^\alpha = b_i^\alpha \sum_\beta a_i^\beta.$$

It is interesting to note that the Poisson algebra of the variables $\boldsymbol{S}_{ij}$ coincides with the defining relations of the Frobenius Lie algebra

$$\{\boldsymbol{S}_{ij}, \boldsymbol{S}_{kl}\} = \delta_{il}(\boldsymbol{S}_{ij} - \boldsymbol{S}_{kj}) + \delta_{jl}(\boldsymbol{S}_{kj} - \boldsymbol{S}_{ij}) + \delta_{jk}(\boldsymbol{S}_{il} - \boldsymbol{S}_{kl}) \tag{4.62}$$

and these relations are compatible with the constraint $\boldsymbol{S}_{ii} = \gamma$.

The appearance of the Frobenius spin variables in the rational ECM model is not accidental. In fact, the same phenomenon takes place for the general elliptic ECM model. To elucidate this fact we recall that the elliptic ECM system is described by the Hamiltonian $H = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{i\neq j} S_{ij}S_{ji}V(q_{ij})$, where $V(q_{ij}) = \mathcal{P}(x)$ is the Weierstrass $\mathcal{P}$-function and S_{ij} are the spin variables defined by (2.7) and having the Poisson bracket

$$\{S_{ij}, S_{kl}\} = \delta_{jk}S_{il} - \delta_{il}S_{kj}.$$

The model is described by the L-operator [8]

$$L = \sum_i (p_i + \zeta(z)S_{ii})E_{ii} + \sum_{i\neq j} \Phi(z, q_{ij})S_{ij}E_{ij}$$

where $\Phi(z,q) = (\sigma(z+q))/(\sigma(z)\sigma(q))$. This L-operator satisfies the Poisson algebra

$$\begin{aligned}\{L_1(z), L_2(w)\} &= [r_{12}(z,w), L_1(z)] - [r_{21}(w,z), L_2(w)] \\ &\quad + \sum_{i\neq j} \frac{\partial}{\partial q_{ij}} \Phi(z-w, q_{ij})(S_{ii} - S_{jj})E_{ij} \otimes E_{ji}\end{aligned} \tag{4.63}$$

with the dynamical r-matrix

$$r_{12}(z,w) = \zeta(z-w)\sum_i E_{ii}\otimes E_{ii} + \sum_{i\neq j} \Phi(z-w, q_{ij})E_{ij}\otimes E_{ji}. \tag{4.64}$$

Due to the last term in (4.63) the model is not integrable. However, the Hamiltonian is invariant under the symmetry $a_i \to k_i a_i$, $b_i \to (1/k_i)b_i$ generated by S_{ii}. The integrability is obtained on the reduced space $S_{ii} = \text{constant}$. As in the case of the RS system, to perform the reduction we define the gauge-invariant L-operator $\boldsymbol{L} = tLt^{-1}$ with $t_{ij} = \delta_{ij}\sum_\alpha a_i^\alpha$ or explicitly

$$\boldsymbol{L} = \sum_i (p_i + \zeta(z))E_{ii} + \sum_{i\neq j} \Phi(z, q_{ij})\boldsymbol{S}_{ij}E_{ij} \tag{4.65}$$

where the gauge-invariant spin variables appear as

$$\boldsymbol{S}_{ij} = S_{ij}\frac{\sum_\alpha a_j^\alpha}{\sum_\alpha a_i^\alpha}$$

computing the Poisson bracket of $\boldsymbol{S}_{ij}$, we find that it precisely coincides with (4.62).

Now it is easy to establish that the Poisson algebra of the L-operator (4.65) has the form

$$\{\boldsymbol{L}_1(z), \boldsymbol{L}_2(w)\} = [\mathbf{r}_{12}(z,w), \boldsymbol{L}_1(z)] - [\mathbf{r}_{21}(w,z), \boldsymbol{L}_2(w)]$$

where a matrix $\mathbf{r}$ literally coincides with the r-matrix of the elliptic Calogero–Moser model [23, 24]

$$\mathbf{r}_{12}(z,w) = (\zeta(z-w) + \zeta(w)) \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \Phi(z-w, q_{ij}) E_{ij} \otimes E_{ji} + \sum_{i \neq j} \Phi(w, q_{ij}) E_{jj} \otimes E_{ij}.$$

Thus, the ECM model corresponds to a representation of the L-operator algebra of the Calogero–Moser model that depends not only on q_i and p_i but also on the additional spin variables $\boldsymbol{S}_{ij}$ with the bracket (4.62). As to the spectral-dependent L-operator of the spin RS model, this does not satisfy the L-operator algebra found for the spinless case [25, 26]. The algebra is quadratic that fixes the form of the corresponding L-operator almost uniquely.

5. Conclusion

In this paper we have presented a detailed description of the Poisson structure for the rational spin RS model by using the Hamiltonian reduction technique. The results obtained cannot be extended to the trigonometric spin RS model in a straightforward manner. It is shown in [12] that the trigonometric RS model can be obtained by means of the Poisson reduction technique applied to the Heisenberg double D associated with $G = GL(N, \mathbb{C})$. Therefore, one may hope to describe the Poisson structure of the trigonometric spin RS model in the same fashion starting from the phase space $D \times G^*$, where G^* is a Poisson–Lie group dual to G.

As is known [5] the rational ECM model possesses the current algebra symmetry and, as we have established, the same symmetry occurs in the rational spin RS model. On the other hand, the trigonometric ECM model has Yangian symmetry. Thus, for the trigonometric spin RS model it is natural to expect the appearance of Yangian symmetry.

The elliptic case is much more involved since at the moment a reduction procedure leading to the elliptic spin RS model is unknown.

Another interesting open problem is to quantize the spin RS models. In the rational case one could use the quantum Hamiltonian reduction procedure developed in [16].

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